J. Fluid Mech. (1981), *vol.* 109, *pp.* 301–309
Printed in Great Britain

Nonlinear dispersive waves in a Hall plasma with a finite conductivity

By A. T. GRANIK
Kentucky State University, Frankfort, Kentucky 40601

(Received 21 July 1980)

The study of nonlinear magnetosonic waves in a turbulent plasma is extended to include the effects of the Hall term. The turbulence and Hall effect are characterized by an effective electrical conductivity and an ion gyrofrequency respectively. It is shown that the magnetosonic waves are governed by a nonlinear equation which can be considered as the generalization of a Korteweg & de Vries (1895) equation with dispersion. For a stationary solution two cases are considered in detail: (*a*) an unperturbed magnetic field is almost parallel to a wave vector, and (*b*) they are almost perpendicular. In the case (*a*) it is shown that the presence of the Hall term can lead to an oscillatory solution which decays due to the finite conductivity. In the second case the Hall effect does not affect the monotonous character of a decaying Taylor-shock profile.

1. Introduction

Nonlinear hydromagnetic waves play an important role in transport processes in a turbulent plasma such as a magnetic neutral sheet (solar flare models, geomagnetic tail), in which effects of perturbations with shorter wavelengths should be taken into account. Therefore it is interesting to study how these effects can influence the behaviour of the turbulent plasma. The turbulence itself is described with the help of an effective electrical conductivity σ_{eff}. A study carried out by Sakai (1972) showed that the finite conductivity leads eventually to the formation of a dissipative hydromagnetic shock wave. The finite ratio of the ion Larmor radius and the perturbation wavelength ($k\rho_L$) in such a plasma can result in either oscillations of these waves (with the Hall effect prevailing over a finite conductivity) or in the flattening of their profiles (with a finite conductivity prevailing over the Hall effect).

In § 2 we show that such waves are described by the equation which is nothing more than the generalization of a Korteweg–de Vries equation with inclusion of a nonlinear dispersive term. In § 2 we study the steady-state solutions of this equation for two cases. First, the unperturbed magnetic field is almost parallel to a wave vector and, second, the unperturbed magnetic field is almost perpendicular to a wave vector. In § 4 we discuss the results of calculations.

2. Derivation of basic equations governing the hydromagnetic waves in a Hall plasma with σ_{eff}

We consider nonlinear hydromagnetic waves propagating in a turbulent plasma with an effective electrical conductivity σ_{eff} (which is present because of micro-instabilities) and a finite ratio of the ion Larmor radius and wavelength comparable to the inverse of the magnetic Reynolds number.

Assuming isothermal character of the plasma we can write the system of MHD equations (Granik 1980*a*):

$$\frac{\partial \rho}{\partial t}+\mathbf{u}\,.\,\nabla\rho+\rho\nabla\,.\,\mathbf{u} = 0, \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\,.\,\nabla\mathbf{u}+(c_s^2/\rho)\nabla\rho-(1/\rho c)\mathbf{j}\times\mathbf{B} = 0, \tag{2}$$

$$\nabla\times\mathbf{E}+(1/c)\frac{\partial \mathbf{B}}{\partial t} = 0, \tag{3}$$

$$\mathbf{E}+(1/c)\mathbf{u}\times\mathbf{B}-(1/\sigma_{\mathrm{eff}})\mathbf{j}-(m_i/c\rho e)\mathbf{j}\times\mathbf{B} = 0, \tag{4}$$

$$\nabla\times\mathbf{B}-(4\pi/c)\mathbf{j} = 0, \tag{5}$$

$$\nabla\,.\,\mathbf{B} = 0, \tag{6}$$

where ρ is the plasma density, $\mathbf{u}$ the velocity of the plasma, c the speed of light, e the electron charge, m_i the ion mass, p the pressure, $\mathbf{B}$ the magnetic field, $\mathbf{j}$ the current density, $c_s^2 = p/\rho$ the constant speed of sound. In the derivation of (4) we neglect the effects of electron inertia and the displacement currents because the ion Larmor radius is much greater than c/ω_{pe} where ω_{pe} is the electron plasma frequency (cf. McKenzie 1971).

The following study of the problem will be carried out in a dimensionless form. For this purpose we introduce the following parameters:

$$\left.\begin{array}{c}\rho = \tilde{\rho}/\rho_0, \quad \omega_i = \tilde{\omega}_i L_0/c_s, \quad t = \tilde{t}c_s/L_0, \quad R_m = 4\pi\sigma_{\mathrm{eff}}c_s L_0/c^2,\\ M_A^2 = \tilde{B}^2/4\pi\rho_0 c_s, \quad \mathbf{u} = \tilde{\mathbf{u}}/c_s, \quad \mathbf{B} = \tilde{\mathbf{B}}/\tilde{B}_0, \quad \tilde{\omega}_i = e\tilde{B}_0/m_i c,\\ \mathbf{r} = \tilde{\mathbf{r}}/L_0.\end{array}\right\} \tag{7}$$

Here the subscript 0 denotes unperturbed constant parameters, superscript tilde means unnormalized parameters, L_0 is the characteristic scale length of the variation of parameters, M_A is the magnetic Mach number, R_m is the magnetic Reynolds number, $\mathbf{r}$ is the radius vector, and $\tilde{\omega}_i$ is the ion gyrofrequency. Performing simple vector operations, we reduce system (1)–(6) with the help of (7) to the following dimensionless form:

$$\frac{\partial \rho}{\partial t}+\mathbf{u}\,.\,\nabla\rho+\rho\nabla\,.\,\mathbf{u} = 0, \tag{8}$$

$$\rho\frac{\partial \mathbf{u}}{\partial t}+\rho\mathbf{u}\,.\,\nabla\mathbf{u}+\nabla\rho+(M_A^2/2)\nabla B^2+M_A^2\mathbf{B}\,.\,\nabla\mathbf{B} = 0, \tag{9}$$

$$\begin{aligned}\frac{\partial \mathbf{B}}{\partial t}&+\mathbf{u}\,.\,\nabla\mathbf{B}-\mathbf{B}\,.\,\nabla\mathbf{u}+\mathbf{B}\nabla\,.\,\mathbf{u}-(M_A^2/\omega_i\rho)\,\mathrm{curl}\,\mathbf{B}\,.\,\nabla\mathbf{B}+(M_A^2/\omega_i\rho)(\mathbf{B}\,.\,\nabla)\,\mathrm{curl}\,\mathbf{B}\\ &-(M_A^2/\omega_i\rho^2)\,\mathrm{curl}\,\mathbf{B}(\mathbf{B}\,.\,\nabla\rho)+(M_A^2/\omega_i\rho^2)\mathbf{B}(\nabla\rho\,.\,\mathrm{curl}\,\mathbf{B})-(1/R_m)\Delta\mathbf{B} = 0,\end{aligned} \tag{10}$$

$$\nabla\,.\,\mathbf{B} = 0. \tag{11}$$

Solutions of system (8)–(11) depend on parameters $\Delta\rho = \rho - 1$ (density fluctuations), $1/R_m$, and $(M_A^2/\omega_i)^2$. They define respectively the nonlinearity of the problem, the dissipation due to the finite conductivity, and the dispersion connected with the finite ion gyrofrequency. We consider such a situation when all the above parameters are of the comparable order of smallness, i.e.

$$\Delta\rho \sim 1/R_m \sim (M_A^2/\omega_i)^2 \sim \epsilon, \tag{12}$$

where ϵ is a smallness parameter. As it was shown by Sakai (1972) the typical values of ϵ for the type IV burst are $10^{-1} \sim 10^{-3}$.

We will study solutions of system (8)–(11) which in the limit of $\epsilon \to 0$ will give characteristic nonlinear waves in an ideally conducting plasma without the Hall currents (Barnes & Hollweg 1974). Therefore we introduce two new independent variables, $\psi(\mathbf{k}.\mathbf{r},\mathbf{t}) = \mathbf{K}.\mathbf{r} - \omega t$, $\tau = \epsilon t$, $\tilde{\mathbf{k}} = \mathbf{k}L_0$, $\omega = \tilde{\omega}L_0/c_s$ are dimensionless wave vector and frequency respectively. Using these new variables, we can rewrite system (8)–(11) (cf. Sakai 1972)

$$\epsilon\frac{\partial\rho}{\partial\tau} - \omega\frac{\partial\rho}{\partial\psi} + (\mathbf{u}\,.\,\mathbf{k})\frac{\partial\rho}{\partial\psi} + \rho\mathbf{k}\,.\frac{\partial\mathbf{u}}{\partial\psi} = 0, \tag{13}$$

$$\epsilon\rho\mathbf{u}_\tau - \rho\omega\mathbf{u}_\psi + \rho(\mathbf{u}\,.\,\mathbf{k})\,\mathbf{u}_\psi + \mathbf{k}\rho_\psi + (M_A^2/2)\,\mathbf{k}B_\psi^2 - M_A^2(\mathbf{B}\,.\,\mathbf{k})\,\mathbf{B}_\psi = 0, \tag{14}$$

$$\begin{aligned}\epsilon\mathbf{B}_\tau - \omega\mathbf{B}_\psi + (\mathbf{u}\,.\,\mathbf{k})\,\mathbf{B}_\psi - (\mathbf{B}\,.\,\mathbf{k})\,\mathbf{u}_\psi + \mathbf{B}(\mathbf{k}\,.\,\mathbf{u}_\psi) - (k^2/R_m)\,\mathbf{B}_{\psi\psi}\\ - (M_A^2/\rho\omega_i)\,(\mathbf{B}\,.\,\mathbf{k})\,(\mathbf{k}\times\mathbf{B}_{\psi\psi}) + (M_A^2/\rho^2\omega_i)\,(\mathbf{B}\,.\,\mathbf{k})\,(\mathbf{k}\times\mathbf{B}_\psi)\,\rho_\psi = 0,\end{aligned} \tag{15}$$

$$\mathbf{k}\,.\,\mathbf{B}_\psi = 0, \tag{16}$$

where

$$\mathbf{u}_\psi = \frac{\partial\mathbf{u}}{\partial\psi},\quad \mathbf{B}_\tau = \frac{\partial\mathbf{B}}{\partial\tau},\quad \mathbf{B}_\psi = \frac{\partial\mathbf{B}}{\partial\psi},\quad \mathbf{B}_{\psi\psi} = \frac{\partial^2\mathbf{B}}{\partial\psi^2},\quad \rho_\tau = \frac{\partial\rho}{\partial t},\quad \rho_\psi = \frac{\partial\rho}{\partial\psi},\quad \mathbf{u}_\tau = \frac{\partial\mathbf{u}}{\partial\tau}.$$

To solve system (13)–(16) we expand all dependent variables around their equilibrium state in power series in ϵ:

$$\rho = 1 + \epsilon\rho_1 + \epsilon^2\rho_2 + \ldots, \tag{17}$$

$$\mathbf{u} = \epsilon\mathbf{u}_1 + \epsilon^2\mathbf{u}_2 + \ldots, \tag{18}$$

$$\mathbf{B} = \mathbf{b}_0 + \epsilon\mathbf{B}_1 + \epsilon^2\mathbf{B}_2 + \ldots, \tag{19}$$

where $\mathbf{b}_0 = \tilde{\mathbf{B}}_0/\tilde{B}_0$. For the subsequent analysis we transform (14) and (15) into a scalar form

$$\epsilon\rho(\mathbf{k}\,.\,\mathbf{u})_\tau - \rho\omega(\mathbf{k}\,.\,\mathbf{u})_\psi + \rho(\mathbf{u}\,.\,\mathbf{k})\,(\mathbf{u}\,.\,\mathbf{k})_\psi + k^2\rho_\psi + M_A^2\,k^2(\mathbf{B}\,.\,\mathbf{B}_\psi) = 0, \tag{20}$$

$$\begin{aligned}\epsilon\rho(\mathbf{b}_0\,.\,\mathbf{u})_\tau - \rho\omega(\mathbf{b}_0\,.\,\mathbf{u})_\psi = \rho(\mathbf{u}\,.\,\mathbf{k})\,(\mathbf{b}_0\,.\,\mathbf{u})_\psi + (\mathbf{b}_0\,.\,\mathbf{k})\,\rho_\psi + M_A^2(\mathbf{b}_0\,.\,\mathbf{k})\,(\mathbf{B}\,.\mathbf{B}_\psi)\\ - M_A^2(\mathbf{B}\,.\mathbf{k})\,(\mathbf{b}_0\,.\,\mathbf{B})_\psi = 0,\end{aligned} \tag{21}$$

$$\begin{aligned}\epsilon(\mathbf{b}_0\,.\,\mathbf{B})_\tau - \omega(\mathbf{b}_0\,.\,\mathbf{B})_\psi + (\mathbf{u}\,.\,\mathbf{k})\,(\mathbf{b}_0\,.\,\mathbf{B})_\psi - (\mathbf{B}\,.\,\mathbf{k})\,(\mathbf{b}_0\,.\,\mathbf{u})_\psi + (\mathbf{b}_0\,.\,\mathbf{B})\,(\mathbf{k}\,.\,\mathbf{u})_\psi\\ - (M_A^2/\rho^2\omega_i)\,(\mathbf{B}\,.\,\mathbf{k})\,[\mathbf{b}_0\,.\,(\mathbf{k}\times\mathbf{B})_\psi] + (M_A^2/\omega_i\rho)\,(\mathbf{B}\,.\,\mathbf{k})\,[\mathbf{b}_0\,.\,(\mathbf{k}\times\mathbf{B})_{\psi\psi}]\\ - (k^2/R_m)\,(\mathbf{b}\,.\,\mathbf{B})_{\psi\psi} = 0,\end{aligned} \tag{22}$$

where, as before, subscripts τ and ψ denote respective derivatives. We need two more equations which can be easily derived from (14) and (15):

$$\epsilon\rho[\mathbf{b}_0.(\mathbf{k}\times\mathbf{u})_\tau]-\rho\omega[\mathbf{b}_0.(\mathbf{k}\times\mathbf{u})_\psi]+\rho(\mathbf{u}.\mathbf{k})[\mathbf{b}_0.(\mathbf{k}\times\mathbf{u})_\psi] - M_A^2(\mathbf{B}.\mathbf{k})[\mathbf{b}_0.(\mathbf{k}\times\mathbf{B})_\psi] = 0, \quad (23)$$

$$\epsilon[\mathbf{b}_0.(\mathbf{k}\times\mathbf{B})_\tau]-\omega[\mathbf{b}_0.(\mathbf{k}\times\mathbf{B})_\psi]+(\mathbf{u}.\mathbf{k})[\mathbf{b}_0.(\mathbf{k}\times\mathbf{B})_\psi] -(\mathbf{B}.\mathbf{k})[\mathbf{b}_0.(\mathbf{k}\times\mathbf{u})_\psi]+(\mathbf{k}.\mathbf{u}_\psi)[\mathbf{b}_0.(\mathbf{k}\times\mathbf{B})]-(M_A^2k^2/\omega_i\rho)(\mathbf{B}.\mathbf{k})(\mathbf{b}_0.\mathbf{B})_{\psi\psi} - k^2/R_m[\mathbf{b}_0.(\mathbf{k}\times\mathbf{B})_{\psi\psi}]+(k^2M_A^2/\rho^2\omega_i)(\mathbf{B}.\mathbf{k})(\mathbf{b}_0.\mathbf{B})_\psi\rho_\psi = 0. \quad (24)$$

It is seen from (16)–(24) that the variables $\mathbf{b}_0.(\mathbf{k}\times\mathbf{B}_\psi)$, $\mathbf{b}_0.(\mathbf{k}\times\mathbf{u})_\psi$ are of the order $\epsilon^{\frac{3}{2}}$ if we take into account (12). This can be explained by the fact that these variables are connected with the dispersive Alfvén wave. In the non-dispersive Alfvén wave they are equal to zero (Granik 1980*b*), and when the dispersion is small ($\sim\epsilon^{\frac{1}{2}}$) they should be of the order $\epsilon^{\frac{3}{2}}$ which follows from (17) and (18). Therefore substituting expressions (17)–(19) into (13), (16), (20)–(22) and equating coefficients of ϵ, $\epsilon^{\frac{3}{2}}$ and ϵ^2 to zero, we obtain sets of equations for the quantities of respective orders. First, for the order ϵ we get

$$\mathbf{A}_0(\mathbf{U}_1)_\psi = 0, \quad (25)$$

where

$$\mathbf{A}_0 = \begin{bmatrix} \omega & -1 & 0 & 0 \\ -k^2 & \omega & 0 & -M_A^2k^2 \\ -(\mathbf{b}_0.\mathbf{k}) & 0 & \omega & 0 \\ 0 & -1 & (\mathbf{b}_0.\mathbf{k}) & \omega \end{bmatrix} \quad (26)$$

and

$$\mathbf{U}_1 = \begin{bmatrix} \rho_1 \\ (\mathbf{k}.\mathbf{U}_1) \\ (\mathbf{b}_0.\mathbf{u}_1) \\ (\mathbf{b}_0.\mathbf{B}_1) \end{bmatrix}. \quad (27)$$

Following the usual procedure we can write solutions of (25) as

$$\mathbf{U}_1 = \mathbf{R}f_1(\psi,\tau), \quad (28)$$

where $\mathbf{R}$ is a column vector satisfying matrix equation

$$\mathbf{A}_0\mathbf{R} = 0. \quad (29)$$

Let us introduce a row vector $\mathbf{L}$ corresponding to a column vector $\mathbf{R}$

$$\mathbf{L}\,\mathbf{A}_0 = 0. \quad (30)$$

Using (26) and (29), we can obtain explicit expressions for vectors $\mathbf{R}$ and $\mathbf{L}$

$$\mathbf{R} = \begin{bmatrix} 1 \\ \omega \\ \mathbf{b}_0.\mathbf{k} \\ \dfrac{\omega^2-k^2}{M_A^2k^2} \end{bmatrix}, \quad \mathbf{L} = \left[1, \frac{\omega}{\omega^2-M_A^2k^2}, \quad -\frac{[(\mathbf{b}_0.\mathbf{k})/\omega]M_A^2k^2}{\omega^2-M_A^2k^2}, \quad \frac{M_A^2k^2}{\omega^2-M_A^2k^2}\right], \quad (31)$$

To find eigenvalues of system (25), we equate det $\mathbf{A}_0$ to zero which gives us the well-known dispersion relation for magnetosonic waves in an ideally conducting plasma without Hall currents (McKenzie 1971)

$$\omega^4 - \omega^2 k^2(1+M_A^2) + M_A^2 k^2 (\mathbf{b}.\mathbf{k})^2 = 0. \tag{32}$$

From the equations corresponding to the order $\epsilon^{\frac{3}{2}}$ we have

$$\mathbf{A}_1(\mathbf{U}_2)_\psi = \mathbf{D}_{1\psi\psi}, \tag{33}$$

where

$$\mathbf{U}_2 = \begin{bmatrix} \mathbf{b}_0.(\mathbf{k}\times\mathbf{u}_1) \\ \mathbf{b}_0.(\mathbf{k}\times\mathbf{B}_1) \end{bmatrix}, \tag{34}$$

$$\mathbf{A}_1 = \begin{bmatrix} \omega & M_A^2(\mathbf{b}_0.\mathbf{k}) \\ (\mathbf{b}_0.\mathbf{k}) & \omega \end{bmatrix}, \tag{35}$$

$$\mathbf{D}_1 = \begin{pmatrix} 0 \\ \dfrac{(M_A k)^2}{\omega_i}(\mathbf{b}_0.\mathbf{k})(\mathbf{b}_0.\mathbf{B}_1) \end{pmatrix}. \tag{36}$$

Because det $\mathbf{A}_1 \neq 0$ we can find using (34)–(36) explicit expressions for elements of vector column $\mathbf{U}_2$ as functions of $\mathbf{b}_0.\mathbf{B}_1$,

$$\mathbf{b}_0.(\mathbf{k}\times\mathbf{B}_1) = -\frac{(M_A k)^2(\mathbf{b}_0.\mathbf{k})^2}{\omega^2-(\mathbf{b}_0.\mathbf{k})^2 M_A^2}(\omega/\omega_i)(\mathbf{b}_0.\mathbf{B}_1)_\psi. \tag{37}$$

Relation (37) permits us to write equations corresponding to the second order in ϵ in the form

$$\mathbf{A}_0(\mathbf{U}_3)_\psi = \mathbf{D}_3, \tag{38}$$

where column vectors $\mathbf{U}_3$ and $\mathbf{D}_3$ are

$$\mathbf{U}_3 = \begin{bmatrix} \rho_2 \\ \mathbf{u}_2.\mathbf{k} \\ \mathbf{b}_0.\mathbf{u}_2 \\ \mathbf{b}_0.\mathbf{B}_2 \end{bmatrix}, \quad \mathbf{D}_3 = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix},$$

and

$$\begin{aligned}
d_1 &= \rho_{1\tau} + (\mathbf{u}_1.\mathbf{k})\rho_{1\psi} + \rho_1(\mathbf{u}.\mathbf{k})_\psi,\\
d_2 &= (\mathbf{u}_1.\mathbf{k})_\tau - \rho_1\omega(\mathbf{u}_1.\mathbf{k})_\psi + (\mathbf{u}_1.\mathbf{k})(\mathbf{u}_1.\mathbf{k})_\psi + \tfrac{1}{2}(M_A k)^2(\mathbf{B}_1^2)_\psi,\\
d_3 &= (\mathbf{u}_1.\mathbf{b}_0)_\tau - \rho_1\omega(\mathbf{b}_0.\mathbf{u}_1)_\psi + (\mathbf{u}_1.\mathbf{k}).(\mathbf{b}_0.\mathbf{u}_1)_\psi + \tfrac{1}{2}M_A^2(\mathbf{b}_0.\mathbf{k})(\mathbf{B}_1^2)_\psi,\\
d_4 &= (\mathbf{b}_0.\mathbf{B}_1)_\tau + (\mathbf{u}_1.\mathbf{k})(\mathbf{b}_0.\mathbf{B}_1)_\psi + (\mathbf{b}_0.\mathbf{B}_1)(\mathbf{u}_1.\mathbf{k})_\psi\\
&\quad - (\omega/\omega^2 - (\mathbf{b}_0.\mathbf{k})M_A^2)(M_A^2 k^2/\omega_i^2)(\mathbf{b}_0.\mathbf{k})(\mathbf{b}_0.\mathbf{B}_1)_{\psi\psi\psi} - (k^2/R_m)(\mathbf{b}_0.\mathbf{B}_1)_{\psi\psi}.
\end{aligned}$$

For the following analysis we will need $\mathbf{B}_1^2 = f(\mathbf{B}_1.\mathbf{b}_0)$. With the help of (16) we write the resolution of vector $\mathbf{B}_1$ along $\mathbf{b}_0\times\mathbf{k}$ and $\mathbf{k}\times(\mathbf{b}_0\times\mathbf{k})$

$$\mathbf{B}_1 = \left(\mathbf{B}_1.\frac{\mathbf{b}_0\times\mathbf{k}}{|\mathbf{b}_0\times\mathbf{k}|}\right)\frac{\mathbf{b}_0\times\mathbf{k}}{|\mathbf{b}_0\times\mathbf{k}|} + (\mathbf{b}_0.\mathbf{B}_1)k^2\frac{\mathbf{k}\times(\mathbf{b}_0\times\mathbf{k})}{|\mathbf{k}\times(\mathbf{b}_0\times\mathbf{k})|^2}. \tag{39}$$

Performing simple vector operations and using (37), we obtain

$$\mathbf{B}_1^2 = (1/k^2-(\mathbf{b}_0.\mathbf{k})^2)\{[(M_A^2 k^4/\omega^2-(\mathbf{b}_0.\mathbf{k})^2 M_A^2)(\omega/\omega_i)(\mathbf{b}_0.\mathbf{B}_1)_\psi]^2+k^2(\mathbf{b}_0.\mathbf{B}_1)^2\}.$$

If we operate on both parts of (38) with the row vector $\mathbf{L}$ and use (30) then the compatibility condition for (25) and (38) gives

$$\mathbf{L}\,\mathbf{D}_3 = 0. \tag{40}$$

Substituting into (39) expressions for elements of the column vector $\mathbf{R}$ from (31) and using (38), we obtain the following equation for the density:

$$\rho_{1\tau}+a\rho_1\rho_{1\psi}+b\rho_{1\psi}\rho_{1\psi\psi}+c\rho_{1\psi\psi}+d\rho_{1\psi\psi\psi} = 0, \tag{41}$$

where a, b, c and d are given by

$$a = \tfrac{1}{2}\omega\frac{3(\omega^2-k^2)+2}{2\omega^2-k^2(1+M_A^2)}, \tag{42}$$

$$b = \tfrac{1}{2}\beta^2\omega^3\frac{k^2(\mathbf{b}_0.\mathbf{k})^2}{[\omega^2-M_A^2(\mathbf{b}_0.\mathbf{k})^2]^2}\frac{\omega^2-k^2}{2\omega^2-k^2(1+M_A^2)},\quad \beta^2 = M_A^2/\omega_i, \tag{43}$$

$$c = -\frac{k^2}{R_m}\frac{\omega^2-k^2}{2\omega^2-k^2(1+M_A^2)}, \tag{44}$$

$$d = -\tfrac{1}{2}\beta^2\omega\frac{k^2(\mathbf{b}_0.\mathbf{k})^2}{\omega^2-(\mathbf{b}_0.\mathbf{k})^2M_A^2}\frac{\omega^2-k^2}{2\omega^2-k^2(1+M_A^2)}. \tag{45}$$

Coefficients a, b, c and d are determined with the help of dispersion equation (32).

Equation (41) is the generalization of a Korteweg–de Vries equation

$$\eta_\tau+\alpha_1\eta\eta_\psi+\alpha_2\eta_{\psi\psi}+\alpha_3\eta_{\psi\psi\psi} = 0. \tag{46}$$

In our equation there is an additional nonlinear term $b\rho_{1\psi}\rho_{1\psi\psi}$ representing a nonlinear effect of a finite ion gyrofrequency on density fluctuations.

3. Solutions of equation (41)

Here we consider only stationary solutions to equation (36). Therefore we introduce the variable

$$z = \psi-M_0\tau, \tag{47}$$

where M_0 is the Mach number of a wave front. Then equation (41) is

$$d\left(\frac{d^2\rho_1}{dz^2}\right)+\tfrac{1}{2}b\left(\frac{d\rho_1}{dz}\right)^2+c\frac{d\rho_1}{dz}+\tfrac{1}{2}a\rho_1^2-M_0\rho_1 = 0, \tag{48}$$

with boundary conditions derived from the assumption of undisturbed upstream and downstream flows (Johnson 1969)

$$\rho_1\to 0,\quad z\to\infty;\quad \rho_1\to 2M_0/a,\quad z\to-\infty. \tag{49}$$

If we use substitution $y = d\rho_1/dz$ then equation (48) is easily transformed into the well-known Abel equation of the second type

$$yy'+(b/2d)y^2+(c/d)y+(a/2d)\rho_1^2-(M_0/d)\rho_1 = 0. \tag{50}$$

The full analysis of the equation (50) with (49) is in preparation. Here we study this equation partially using results of the continuing work.

At first, we consider $\mathbf{b}_0 \parallel \mathbf{k}$. From dispersion equation (32) it follows that

$$\omega^2 - M_A^2 k^2 = 0.$$

But this is nothing more than the Alfvén wave where according to (37) $\mathbf{B}_1 \perp \mathbf{b}_0$ and hence $\rho_1 = 0$. Non-trivial solutions exist only for $\mathbf{B}_1$ and $\mathbf{u}_1$. Using this result it is possible to consider the case of a fast magnetoacoustic wave with $\mathbf{b}_0$ almost parallel to $\mathbf{k}$.

From (32) we obtain

$$\omega^2 = k^2 M_A^2 [1 + \alpha/(M_A^2 - 1) + O(\alpha^2)] \tag{51}$$

where $\alpha = [1 - (\mathbf{b}_0 . \mathbf{k})^2/k^2] \ll 1$. If $(\mathbf{B}_1 . \mathbf{b}_0) \sim \rho_1$ is of the order α^2 and $\alpha \sim \beta^2$ which is compatible with the case of Alfvén waves (with the analogous scaling of the orders of smallness) then we can linearize (48):

$$\frac{d^2\rho_1}{dz^2} + (c/d)\frac{d\rho_1}{dz} - (M_0/d)\rho_1 = 0, \tag{52}$$

where

$$\gamma \equiv \frac{d}{c} = (k/M_A)\frac{M_A^2 - 1}{M_A^2}(\beta^2/\alpha)(1/R_m). \tag{53}$$

If $M_A^2 - 1 > 0$ then $d < 0$. Therefore for $c^2 - 4M_0|d| < 0$ we have exponential damping of the oscillating density, rate of which is determined by γ. If $M_A^2 < 1$ then $d > 0$ and $c < 0$, which means that for bounded ρ_1 we have a pure damping. Analogous result is true for the case $M_A^2 - 1 > 0$ $(d < 0, c < 0)$ and $c^2 - 4M_0|d| > 0$. These conclusions are in full agreement with the phase-plane analysis of (36) which shows the existence of a singular spiral point (or saddle, or node points) depending on the sign of the expression $c_2 - 4M_0|d|$.

Because the above case does not show the formation of a shock wave, it is interesting to study the following case: $\mathbf{b}_0$ is almost perpendicular to $\mathbf{k}$. From (32) we have for the fast magnetoacoustic wave

$$\omega^2 = k^2(1 + M_A^2)[1 - \alpha_4 M_A^2/k^2(1 + M_A^2)],$$

where $\alpha_4 = (\mathbf{b}_0 . \mathbf{k})^2$. We consider only the fast mode because this mode is the only one which does not undergo Landau damping in the direction normal to $\mathbf{k}$ (Barnes 1966). As it is seen from the expressions (42)–(45), terms connected with the dispersion (b and d) are of the order α_4 as compared with the terms connected with the 'pure' nonlinearity (a) and dissipation (c). Therefore we can expand the solution of (48) in power series in α_4

$$\rho_1 = \rho_{10} + \alpha_4 \rho_{11} + \dots. \tag{54}$$

Then for the zeroth order we obtain (cf. Sakai 1972)

$$\rho_{10} = (M_0/a)\left[1 - \tanh\left(\frac{M_0 Z}{2|c|}\right)\right]. \tag{55}$$

For the first order we have

$$\rho_{11} = -\beta^2 \frac{kM_A^2}{2(1+M_A^2)^{\frac{3}{2}}}\frac{M_0^3}{ac^2}\left[\cosh^{-2}\left(\frac{M_0 z}{2|c|}\right)\ln\cosh\left(\frac{M_0 z}{2|c|} - \frac{M_0}{4a}\tanh\left(\frac{M_0 z}{2|c|}\right)\right]. \tag{56}$$

Here

$$c \cong -(k^2/R_m)(M_A^2/(1+M_A^2)) + O(\alpha_4),$$

$$a \cong \tfrac{1}{2}k(1+M_A)^{\frac{1}{2}} \frac{3k^2M_A^2+2}{k^2(1+M_A^2)} + O(\alpha_4^2).$$

We used in derivation of (54) condition

$$\rho_1(0) = M_0/a.$$

As it is seen from (54) in the case of propagation of a shock wave in the Hall plasma there is correctional term (56) which tends to flatten the profile of the hydromagnetic shock wave but does not change its monotonous character. These facts are also in a full agreement with a phase-plane analysis of (36). The shock-wave front thickness $\mathscr{L}$ is determined by the following relation

$$\mathscr{L} = \tilde{\mathscr{L}}/L_0 \sim (M_0/2|c|)^{-1}(1+s), \tag{57}$$

where $\mathscr{L}$ is unnormalized front thickness,

$$s = \alpha_4 \beta^2 \frac{M_0^2}{ac^2} \frac{kM_A^2}{8(1+M_A^2)^{\frac{3}{2}}}$$

and $(M_0/2|c|)^{-1}$ is nothing more than the front thickness for conventional plasma (Sakai 1972).

4. Discussion

Sakai suggested that the effects of a finite ion gyrofrequency can lead to the dispersive effects in nonlinear MHD waves in a plasma with a finite electrical conductivity. His guess that these effects would introduce an additional term $d\rho_{\psi\psi\psi}$ into (35) was correct. Moreover, we show that there is one more term $b\rho_{1\psi}\rho_{1\psi\psi}$ representing interaction of the plasma density and the pressure variation connected with the Hall effect.

It is shown in the present study that the consideration of the finite ion gyrofrequency in a turbulent plasma with a finite conductivity σ_{eff} can lead to damped oscillations of all quantities. In the case of an average magnetic field normal to the direction of propagation, the above effects result in flattening of a Taylor-shock profile. We now estimate the shock-front thickness using (55) and (56):

$$\tilde{\mathscr{L}}_{\text{fast}} \sim (c^2/\tilde{\omega}_{\text{pe}}^2) \frac{\tilde{\nu}_{\text{eff}}}{\tilde{V}_0} \frac{\tilde{V}_A^2}{c_s^2 + \tilde{V}_A^2}(1+s),$$

where ν_{eff} is the effective collision frequency defined by the relation $\sigma_{\text{eff}} = \omega_{\text{pe}}^2/4\pi\nu_{\text{eff}}$ Sakai 1972), $\tilde{V}_0$ is the front velocity, and $\tilde{V}_A$ is the Alfven velocity. Here we used the estimation $k \sim 1$. From the above expression we can see that the front thickness is greater by factor $(1+s)$ compared with the usual hydromagnetic shock wave.

The author would like to express his gratitude to Mrs A. Casner and Mrs J. Clark for help rendered during the preparation of typing of this manuscript.

REFERENCES

BARNES, A. & HOLLWEG, J. 1974 *J. Geophys. Res.* **75**, 2302.
GRANIK, A. 1980*a* *J. Plasma Physics* **24**, 213.
GRANIK, A. 1980*b* *Proc. Int. Conf. on Nonlinear Phenomena, Arlington, Texas* (to be published).
JOHNSON, R. 1969 *J. Fluid Mech.* **42**, 45.
KORTEWEG, D. & VRIES, G. DE 1895 *Phil. Mag.* **29** (5), 422.
MCKENZIE, J. 1971 *J. Plasma Phys.* **5**, 275.
SAKAI, J.-I. 1972 *Cosmic Electrodynamics* **3**, 260.